Mappings between R-tors *and other lattices.*

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Notation:

- R will denote an associative ring with identity.
- R-Mod will denote the category of unital left R-modules.
- $\mathrm{S}(M)$, the complete lattice of submodules of M.
- $\mathrm{S}_{fi}(M)$, the lattice of fully invariant submodules of M.
- R-simp a complete set of representatives of isomorphism classes of simple modules

A *preradical* r for $\mathrm{R\text{-}Mod}$ is a subfunctor of the identity functor for $\mathrm{R\text{-}Mod}$, that is, for every module homomorphism $f : M \to N$ it happens that $f(r(M)) \subseteq r(N)$.

R-pr :

The class of all preradicals in $\mathrm{R\text{-}Mod}$.

For each $M \in \mathrm{R\text{-}Mod}$ and $r, s \in \mathrm{R\text{-}pr}$,

- Order: $r \preceq s$ if $r(M) \leq s(M)$ for each $M \in \mathrm{R\text{-}Mod}$.
- Meet: $(r \wedge s)(M) = r(M) \cap s(M)$,
- Join $(r \vee s)(M) = r(M) + s(M)$.
- Product: $(r \cdot s)(M) = r(s(M))$.
- Coproduct: $(r : s)(M)$ is such that $(r : s)(M)/r(M) = s(M/r(M))$.

The lattice structure of $\mathrm{R}\text{-pr}$

$\mathrm{R}\text{-pr}$ with the partial ordering $\preceq$, is a complete, atomic, coatomic, modular, upper continuous and strongly pseudocomplemented big lattice

The least element is the zero functor denoted by $\underline{0}$ and the identity functor $\underline{1}$ is the greatest element.

- The class of atoms of $\mathrm{R}\text{-pr}$ is $\{\alpha_S^{E(S)} \mid S \in \mathrm{R}\text{-simp}\}$.
- The class of coatoms of $\mathrm{R}\text{-pr}$ is $\{\omega_I^{\mathrm{R}} \mid I$ is a maximal two sided ideal of $\mathrm{R}\}$.

For $r \in \mathrm{R\text{-}pr}$, we say that

- r is idempotent if $r \cdot r = r$.
- r is a radical if $r : r = r$.
- r is a left exact preradical if $r(N) = N \cap r(M)$ for every $N \leq M$ and $M \in \mathrm{R\text{-}Mod}$.
- r is a $t-$radical when $r(M) = r(R)M$ for every $M \in \mathrm{R\text{-}Mod}$.

R-id	R-rad
idempotent preradicals	radicals

R-lep	R-trad
left exact preradicals	t-radicals

R-ler	R-rid
left exact radicals	idempotent radicals

For each $r \in \mathrm{R\text{-}pr}$, we denote

- $\mathbb{T}_r = \{M \mid r(M) = M\}$, *the pretorsion class associated to* r,
- $\mathbb{F}_r := \{M \mid r(M) = 0\}$, *the pretorsion free class associated to* r.

Alpha and omega preradicals.

- For $N \in \mathrm{S}_{fi}(M)$, there are two distinguished preradicals, α_N^M and ω_N^M, assigning M to N, which are defined as follows:

$$\alpha_N^M(L) := \sum \{f(N) \mid f \in \mathrm{Hom_R}(M, L)\}$$

$$\omega_N^M(L) := \cap \{f^{-1}(N) \mid f \in \mathrm{Hom_R}(L, M)\},$$

for each $L \in \mathrm{R\text{-}Mod}$.

- If N is a fully invariant submodule of M, we have that the class $\{r \in \mathrm{R\text{-}pr} \mid r(M) = N\}$ is precisely the interval $[\alpha_N^M, \omega_N^M]$.

A *torsion theory* for $\mathrm{R}\text{-}\mathrm{Mod}$ is an ordered pair $(\mathbb{T}, \mathbb{F})$ of classes of modules such that:

(i) $\mathrm{Hom}(T, F) = 0$ for every $T \in \mathbb{T}$ and for every $F \in \mathbb{F}$.

(ii) If $\mathrm{Hom}_{\mathrm{R}}(C, F) = 0$ for all $F \in \mathbb{F}$, then $C \in \mathbb{T}$.

(iii) If $\mathrm{Hom}_{\mathrm{R}}(T, C) = 0$ for all $T \in \mathbb{T}$, then $C \in \mathbb{F}$.

Notice that

- $\mathbb{T}$ is a torsion class (i.e a class closed under taking quotients, direct sums and extensions)
- $\mathbb{F}$ is a torsion-free class (i.e a class closed under taking submodules, direct products and extensions).
- $(\mathbb{T}, \mathbb{F})$ is a hereditary torsion theory if and only if $\mathbb{T}$ is closed under submodules. If in adition $\mathbb{F}$ is closed under quotients, $(\mathbb{T}, \mathbb{F})$ is a cohereditary and hereditary torsion theory.

R − TORS

The big lattice of all torsion theories in R-Mod.

R-tors

The frame of all hereditary torsion theories in R-Mod.

R-qtors

All hereditary torsion theories which are cohereditary

- For $\tau \in \mathrm{R\text{-}tors}$, we shall write $\mathbb{T}_\tau$ for the τ-torsion class and $\mathbb{F}_\tau$ for the τ-torsion free class, so that $\tau = (\mathbb{T}_\tau, \mathbb{F}_\tau)$.
- For a class $\mathcal{A}$ of modules:
  - $\boldsymbol{\xi(\mathcal{A})}$ is **the hereditary torsion theory generated by** $\mathcal{A}$.
  - $\boldsymbol{\chi(\mathcal{A})}$ is **the hereditary torsion theory cogenerated by** $\mathcal{A}$**.**
- We shall write ξ (respectively, χ) for the least (resp., greatest) element of $\mathrm{R\text{-}tors}$.
- For $\tau \in \mathrm{R\text{-}tors}$, let $\tau^\perp$ stand for the pseudocomplement in $\mathrm{R\text{-}tors}$ of τ.

Mappings beween R-tors and R-pr.

Well Known Facts:

- There exist lattice isomorphisms

$$\varphi : \text{R-rid} \longrightarrow \text{R-TORS},$$

$$\zeta : \text{R-ler} \longrightarrow \text{R-tors}$$

  where both are given by $r \mapsto (\mathbb{T}_r, \mathbb{F}_r)$.

- In both cases, the inverse is $(\mathbb{T}, \mathbb{F}) \mapsto t_{(\mathbb{T},\mathbb{F})}$, where $t_{(\mathbb{T},\mathbb{F})} = \bigvee\{\alpha_T^T \mid T \in \mathbb{T}\}$. (Note that $t_{(\mathbb{T},\mathbb{F})}$, the so-called *torsion part*, coincides with $\bigwedge\{\omega_0^F \mid F \in \mathbb{F}\}$.

- There exists a canonical isomorphism

$$\eta : \mathrm{S}_{fi}(\mathrm{R}) \to \mathrm{R}\text{-trad},$$

$\eta(I) := \alpha_I^{\mathrm{R}}$ (which is, observe, left multiplication by I). Its inverse sends $r \mapsto r(\mathrm{R})$.

Define a mapping

$$t : \mathrm{R\text{-}tors} \longrightarrow \mathrm{R\text{-}pr}, \quad \tau \mapsto t_\tau.$$

Remark

1. The mapping t is always injective, order-preserving and preserves infima. This also holds in the arbitrary case.
2. In general, t **does not preserve suprema.** Indeed, while it is true, for $\tau, \sigma \in \mathrm{R\text{-}tors}$, that $t_\tau \vee t_\sigma \preceq t_{\tau \vee \sigma}$, equality does not always hold.

Example

Let R be the subring of $\mathbb{Z}_2^{\mathbb{N}}$ spanned by 1 and $\mathbb{Z}_2^{(\mathbb{N})}$, so that R consists of sequences of zero and ones eventually constant. Denote as Z the left exact preradical sending each module to its singular submodule. It can be proved that:

1. Every simple ideal is a direct summand of R, and therefore $\mathrm{soc}_p(\mathrm{R}) = \mathrm{soc}(\mathrm{R}) = \mathbb{Z}_2^{(\mathbb{N})}$.
2. $\mathbb{Z}_2^{(\mathbb{N})} \leq_e \mathrm{R}$.
3. $\mathbb{T}_{\tau_G} = \mathbb{F}_{\tau_{\mathcal{SP}}}$, and $\tau_G \in \mathrm{R}\text{-jans}$.
4. $\tau_{\mathcal{SP}}^{\perp} = \tau_G$,
5. $\tau_{\mathcal{SP}} \vee \tau_G = \tau_{\mathcal{SP}} \vee \tau_{\mathcal{SP}}^{\perp} = \xi(\mathcal{SP}) \vee \chi(\mathcal{SP}) = \chi$.
6. Thus, $t_{\tau_{\mathcal{SP}} \vee \tau_G}(\mathrm{R}) = \mathrm{R}$, but $t_{\tau_{\mathcal{SP}}}(\mathrm{R}) + t_{\tau_G}(\mathrm{R}) = soc_p(\mathrm{R}) = \mathbb{Z}_2^{(\mathbb{N})} \neq \mathrm{R}$.

Lemma

Let $\{\tau_i\}_{i\in I} \subseteq \mathrm{R\text{-}qtors}$*. Then* $t_{\bigvee_{i\in I}\tau_i} = \bigvee_{i\in I} t_{\tau_i}$*, taking the supremum on the left in* R-tors *and the one on the right in* R-pr.

Thus, over any left perfect ring, R-qtors is a complete sublattice both of R-tors and (via a canonical embedding) of R-pr.

Theorem

Let R *be a left perfect ring. Then* R-qtors *is a complete sublattice of* R-tors *and* $t_{|_{\text{R-qtors}}} : \text{R-qtors} \longrightarrow \text{R-pr}$ *is a complete lattice embedding.*

Mappings between R-tors and $S_{fi}(R)$

Define a mapping, *evaluation* (at the ring),

$$e : \text{R-tors} \longrightarrow \mathrm{S}_{fi}(\mathrm{R}),$$
$$\tau \mapsto t_\tau(R).$$

Remark

Notice that e preserves orderings and arbitrary infima.
In general, e **does not preserve even binary suprema.**

- A module $M \in \mathrm{R\text{-}Mod}$ is called a *Kasch module* if and only if every $S \in \mathrm{R\text{-}simp}$ is embeddable in M.
- R is a left Kasch ring if $E(R)$ is an injective cogenerator for $\mathrm{R\text{-}Mod}$.

Proposition

If the mapping e is injective, then R *is a left Kasch ring.*

Recall that a ring R is said to be *left fully idempotent*(or *left weakly regular*) when every left ideal is idempotent.

Proposition

If R *is left fully idempotent and e is injective, then* R *is a left Bronowitz-Teply ring (i.e.* $R\text{-qtors} = R\text{-tors}.$*)*

Lemma

If $R\text{-trad} = R\text{-ler}$, *then* R *is a semisimple ring.*

Proposition

If R *is regular (in the sense of von Neumann) and e is an isomorphism, then* R *is semisimple.*

For an arbitrary ring R, abbreviate as $\mathscr{E}(R)$ the statement "$e : \mathrm{R}\text{-tors} \to \mathrm{S}_{fi}(\mathrm{R})$ is a lattice isomorphism".

Theorem

Let R be a ring such that $\mathscr{E}(\mathrm{R})$ and ${}_{\mathrm{R}}\mathcal{SP}$ is finite. Then $\mathrm{R} = \mathrm{R}_s \times \mathrm{R}_e$ for some (possibly trivial) semisimple ring R_s and some ring R_e such that $\mathscr{E}(\mathrm{R}_e)$ and $\mathrm{soc}_p(\mathrm{R}_e) = 0$.

Let us now consider the inverse of e, when e is a lattice isomorphism. Set

$$\alpha,\ \beta : \mathrm{S}_{fi}(\mathrm{R}) \to \text{R-tors}$$

$$\alpha(I) := \xi(I) = \bigwedge\{\tau \in \text{R-tors} \mid I \in \mathbb{T}_\tau\}$$

and

$$\beta : \mathrm{S}_{fi}(\mathrm{R}) \to \text{R-tors}$$

$$\beta(I) := \chi(\mathrm{R}/I) = \bigvee\{\tau \in \text{R-tors} \mid R/I \in \mathbb{F}_\tau\},$$

for each $I \in \mathrm{S}_{fi}(\mathrm{R})$.

Theorem

If any one of α, β or e is a lattice isomorphism, then all three are isomorphisms, $\alpha = \beta$ and its inverse is e.

Mappings between $\mathrm{S}_{fi}(M)$ and R-ler

Now, we continue the previous study for any $M \in \mathrm{R\text{-}Mod}$ and $\mathrm{S}_{fi}(M)$.

- For $L, K \in \mathrm{S}_{fi}(M)$, consider the product

$$K_M L := \alpha_K^M(L)$$

- We say that M is *fully idempotent* if and only if $N_M N = N$ for every $N \in \mathrm{S}_{fi}(M)$.
- Notice that R is fully idempotent if and only if every two-sided ideal is idempotent.

Set,

$$\lambda_M : \mathrm{S}_{fi}(M) \to \text{R-pr}$$

$$\lambda_M(N) = \alpha_N^M$$

and

$$e_M : \text{R-tors} \to \mathrm{S}_{fi}(M)$$

$$e_M(\tau) := t_\tau(M)$$

It is clear that $e_{\mathrm{R}} = e$.
Taking the isomorphism $\zeta : \text{R-ler} \to \text{R-tors}$, notice that $e_M \circ \zeta : r \mapsto r(M)$.

Proposition

Let $M \in \mathrm{R}\text{-Mod}$ be such that the assignation $\lambda_M : \mathrm{S}_{fi}(M) \to \mathrm{R}\text{-ler}$ given by $\lambda_M(N) = \alpha_N^M$ is well-defined and a lattice isomorphism. Then, the following conditions hold.

(a) $\mathrm{S}_{fi}(M)$ *is an atomic frame.*

(b) M *is fully idempotent.*

(c) $e_M \circ \zeta$ *is the inverse of* λ_M.

(d) M *is a generator for* $\mathrm{R}\text{-Mod}$.

(e) M *is a Kasch module,* $t_{\xi(S)}(M) = \mathrm{soc}_S(M)$ *for every* $S \in \mathrm{R}\text{-simp}$, $\mathrm{soc}(M)$ *is the least essential element of* $\mathrm{S}_{fi}(M)$, *and* $e_M(\tau_D) = \mathrm{soc}(M)$.

(f) *For every* $\tau \in \mathrm{R}\text{-tors}$, $t_\tau = \alpha^M_{t_\tau(M)}$.

(g) *For every* $I \in \mathrm{S}_{fi}(\mathrm{R})$, $\alpha^R_I = \alpha^M_{IM}$.

(h) *If M is projective, then, for every* $\tau \in \mathrm{R}\text{-tors}$, $t_\tau = \alpha^{\mathrm{R}}_{t_\tau(\mathrm{R})}$.

Theorem

For a ring R*, the following statements are equivalent.*

(a) R *is semisimple.*

(b) R-trad $=$ R-ler.

(c) *For every projective generator* P, $\lambda_P : \mathrm{S}_{fi}(P) \to$ R-ler *is well-defined and a lattice isomorphism.*

(d) $\lambda_{\mathrm{R}} : \mathrm{S}_{fi}(\mathrm{R}) \to$ R-ler *is well-defined and a lattice isomorphism.*

(e) *There is some projective module* P *such that* $\lambda_P : \mathrm{S}_{fi}(P) \to$ R-ler *is well-defined and a lattice isomorphism.*

Thank you!

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